

Maximum Mass-Radius Ratio for Compact General Relativistic Objects in Schwarzschild- de Sitter Geometry

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Abstract

Upper limits for the mass-radius ratio are derived for arbitrary general relativistic matter distributions in the presence of a cosmological constant. General restrictions for the red shift and total energy (including the gravitational contribution) for compact objects in the Schwarzschild-de Sitter geometry are also obtained in terms of the cosmological constant and of the mean density of the star.

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I. INTRODUCTION

The possibility that the cosmological constant be nonzero and dominates the energy density of the Universe today is one of the most intriguing problems of the contemporary

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physics. Data recently collected by two survey teams (the Supernova Cosmology Project [1] and the High- z Supernova Search Team [2]) and analyzed in the framework of homogeneous FLRW cosmological models, have yielded, as a primary result, a strictly positive cosmological constant of order unity.

Within the classical GR, the existence of a cosmological constant is equivalent to the postulate that the total energy momentum tensor of the Universe $T_{ik}^{(U)}$ possesses an additional piece $T_{ik}^{(V)}$, besides that of its matter content $T_{ik}^{(m)}$, of the form $T_{ik}^{(V)} = \Lambda g_{ik}$, where generally the cosmological constant Λ is a scalar function of space and time. Such a form of the additional piece has previously been obtained in certain field - theoretical models and is interpreted as a vacuum contribution to the energy momentum tensor [3-5], $\Lambda g_{ik} = \langle T_{ik}^{(V)} \rangle = \frac{8\pi G}{c^4} \langle \rho_V \rangle g_{ik}$, where ρ_V is the energy of the vacuum. The vacuum value of T_{ik} thus appears in the form of a cosmological constant in the gravitational field equations (for a review of the cosmological constant problem see [6]).

The cosmological constant can also be interpreted as a parameter measuring the intrinsic temperature of the empty space-time, or in a sense, of the geometry itself [7].

At interplanetary distances, the effect of the cosmological constant could be imperceptible. However, Cardona and Tejeiro [8] have shown that a bound of this constant can be obtained using the values observed from the Mercury's perihelion shift in a Schwarzschild - de Sitter space-time. The presence of a cosmological constant implies that, for the first time after inflation, in the present epoch its role in the dynamics of the Universe becomes dominant. On the other hand there is the possibility that scalar fields presents in the early Universe could condense to form the so called boson stars [9-10]. There are also suggestions that the dark matter could be made up of bosonic particles. This bosonic matter would condense through some sort of Jeans instability to form compact gravitating objects. A boson star can have a mass comparable to that of a neutron star [11]. The simplest kind of boson star is made up of a self - interacting complex scalar field Φ describing a state of zero temperature [12-13]. The self-consistent coupling of the scalar field to its own gravitational

field is via the Lagrangian $L = \frac{1}{2k}\sqrt{-g}R + \frac{1}{2}\sqrt{-g}\left[g^{ik}\Phi_{,i}^*\Phi_{,k} - V(|\Phi|^2)\right]$ where $V(|\Phi|^2)$ is the self-interaction potential usually taken in the form $V(|\Phi|^2) = \frac{1}{2}m^2|\Phi|^2 + \frac{1}{4}\lambda|\Phi|^4$. m is the mass of the scalar field particle (the boson) and λ is the self-interaction parameter. For the bosonic field the stationarity ansatz is assumed, $\Phi(t, r) = \phi(r)e^{-i\omega t}$. If we suppose that in the star's interior regions and for some field configurations the scalar field is constant then in the gravitational field equations the scalar field self-interaction potential will play the role of a cosmological constant, which could also describe a mixture of ordinary matter and bosonic particles.

By using the static spherically symmetric gravitational field equations Buchdahl [14] has obtained an absolute constraint of the maximally allowable mass M - radius R ratio for isotropic fluid spheres of the form $\frac{2M}{R} < \frac{8}{9}$ (we use natural units $c = G = 1$). It is the purpose of the present Letter to investigate the maximum allowable mass -radius ratio in the case of compact general relativistic objects in the presence of a cosmological constant and to study the possible effects of the cosmological constant upon the red-shift and total energy of general relativistic compact objects.

II. MAXIMUM MASS-RADIUS RATIO FOR COMPACT OBJECTS IN SCHWARZSCHILD- DE SITTER GEOMETRY

For a static general relativistic spherically symmetric matter configuration with interior line element given by $ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ the components of the energy-momentum tensor are $T_0^0 = \rho$, $T_1^1 = T_2^2 = T_3^3 = -p$, where ρ is the energy density and p the thermodynamic pressure.

In the presence of a cosmological constant the properties of a compact object can be described completely by the gravitational structure equations, which are given by:

$$\frac{dm}{dr} = 4\pi\rho r^2, \tag{1}$$

$$\frac{dp}{dr} = -\frac{(\rho + p)\left[m + 4\pi\left(p - \frac{2\Lambda}{3}\right)r^3\right]}{r^2\left(1 - \frac{2m}{r} - \frac{8\pi}{3}\Lambda r^2\right)}, \tag{2}$$

$$\frac{d\nu}{dr} = \frac{2 \left[m + 4\pi \left(p - \frac{2\Lambda}{3} \right) r^3 \right]}{r^2 \left(1 - \frac{2m}{r} - \frac{8\pi}{3} \Lambda r^2 \right)}, \quad (3)$$

where $m(r)$ is the mass inside radius r .

Eqs. (1)-(3) must be considered together with an equation of state of the dense matter, $p = p(\rho)$ and with the boundary conditions $p(R) = 0$, $p(0) = p_c$ and $\rho(0) = \rho_c$, where ρ_c and p_c are the central density and pressure, respectively.

With the use of Eqs. (1)-(3) it is easy to show that the function $\zeta = e^{\frac{\nu}{2}} > 0, \forall r \in [0, R]$ obeys the equation

$$\sqrt{1 - \frac{2m(r)}{r} - \frac{8\pi}{3} \Lambda r^2} \frac{1}{r} \frac{d}{dr} \left[\sqrt{1 - \frac{2m(r)}{r} - \frac{8\pi}{3} \Lambda r^2} \frac{1}{r} \frac{d\zeta}{dr} \right] = \frac{\zeta}{r} \frac{d}{dr} \frac{m(r)}{r^3}. \quad (4)$$

Since the density ρ does not increase with increasing r , the mean density of the matter $\langle \rho \rangle = \frac{3m(r)}{4\pi r^3}$ inside radius r does not increase either. Therefore we assume that inside a compact general relativistic object the condition $\frac{d}{dr} \frac{m(r)}{r^3} < 0$ holds independently of the equation of state of dense matter.

Introducing a new independent variable $\xi = \int_0^r r' \left(1 - \frac{2m(r')}{r'} - \frac{8\pi}{3} \Lambda r'^2 \right)^{-\frac{1}{2}} dr'$ [13], from Eq.(4) we obtain the basic result that all stellar type general relativistic matter distributions with negative density gradient obey the condition

$$\frac{d^2 e^{\frac{\nu(\xi)}{2}}}{d\xi^2} < 0, \forall r \in [0, R]. \quad (5)$$

Using the mean value theorem we conclude $\frac{de^{\frac{\nu(\xi)}{2}}}{d\xi} \leq \frac{e^{\frac{\nu(\xi)}{2}} - e^{\frac{\nu(0)}{2}}}{\xi}$, or, taking into account that $e^{\frac{\nu(0)}{2}} > 0$ we find $\frac{de^{\frac{\nu(\xi)}{2}}}{d\xi} \leq \frac{e^{\frac{\nu(\xi)}{2}}}{\xi}$. In the initial variables we have

$$\frac{m(r) + 4\pi \left(p - \frac{2\Lambda}{3} \right) r^3}{r^3 \sqrt{1 - \frac{2m}{r} - \frac{8\pi}{3} \Lambda r^2}} \leq \left[\int_0^r r' \left(1 - \frac{2m(r')}{r'} - \frac{8\pi}{3} \Lambda r'^2 \right)^{-\frac{1}{2}} dr' \right]^{-1}. \quad (6)$$

Since for stable stellar type compact objects $\frac{m}{r^3}$ does not increase outwards $\frac{m(r')}{r'} \geq \frac{m(r)}{r} \left(\frac{r'}{r} \right)^2, \forall r' \leq r$. We denote $\alpha(r) = 1 + \frac{4\pi}{3} \Lambda \frac{r^3}{m(r)}$. Moreover, we assume that in the presence of a cosmological constant the condition $\frac{\alpha(r')m(r')}{r'} \geq \frac{\alpha(r)m(r)}{r} \left(\frac{r'}{r} \right)^2$, or, equivalently,

$$\left(1 + \frac{4\pi}{3} \Lambda \frac{r^3}{m(r')} \right) \frac{m(r')}{r'} \geq \left(1 + \frac{4\pi}{3} \Lambda \frac{r^3}{m(r)} \right) \frac{m(r)}{r} \left(\frac{r'}{r} \right)^2, \quad (7)$$

holds inside the compact object. In fact Eq. (7) is independent of the cosmological constant Λ and is valid for all decreasing density compact matter distributions.

Therefore we can evaluate the RHS of Eq. (6) as follows:

$$\begin{aligned} \int_0^r \frac{r'}{\left(1 - \frac{2m(r')}{r'} - \frac{8\pi}{3}\Lambda r'^2\right)^{\frac{1}{2}}} dr' &= \int_0^r \frac{r'}{\left(1 - \frac{2\alpha(r')m(r')}{r'}\right)^{\frac{1}{2}}} dr' \geq \\ \int_0^r \frac{r'}{\left[1 - \frac{2\alpha(r)m(r)}{r} \left(\frac{r'}{r}\right)^2\right]^{\frac{1}{2}}} dr' &= \frac{r^3}{2\alpha(r)m(r)} \left[1 - \left(1 - \frac{2\alpha(r)m(r)}{r}\right)^{\frac{1}{2}}\right] \end{aligned} \quad (8)$$

With the use of Eq. (8), Eq.(6) becomes:

$$\frac{m(r) + 4\pi \left(p - \frac{2\Lambda}{3}\right) r^3}{\sqrt{1 - \frac{2m(r)}{r} - \frac{8\pi}{3}\Lambda r^2}} \leq \frac{2m(r) \left(1 + \frac{4\pi}{3}\Lambda \frac{r^3}{m(r)}\right)}{1 - \sqrt{1 - \frac{2m(r)}{r} - \frac{8\pi}{3}\Lambda r^2}}. \quad (9)$$

Eq. (9) is valid for all r inside the star. It does not depend on the sign of Λ .

Consider first the case $\Lambda = 0$. By evaluating (9) for $r = R$ we obtain $\frac{1}{\sqrt{1 - \frac{2M}{R}}} \leq 2 \left[1 - \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}\right]^{-1}$, leading to the well-known result $\frac{2M}{R} \leq \frac{8}{9}$ [13].

For $\Lambda \neq 0$, Eq. (9) leads to the following upper limit for mass-radius ratio of compact objects

$$\frac{2M}{R} \leq \left(1 - \frac{8\pi}{3}\Lambda R^2\right) \left[1 - \frac{1}{9} \frac{\left(1 - \frac{2\Lambda}{\bar{\rho}}\right)^2}{1 - \frac{8\pi}{3}\Lambda R^2}\right], \quad (10)$$

where $\bar{\rho} = \frac{3M}{4\pi R^3}$ is the mean density of the star.

In order to find a general restriction for $\bar{\rho}$ we shall consider the behavior of the Ricci invariant $r_2 = R_{ijkl}R^{ijkl}$. If the static line element is regular, satisfying the conditions $e^{\nu(0)} = \text{const.} \neq 0$ and $e^{\lambda(0)} = 1$, then the Ricci invariants are also non-singular functions throughout the star. In particular for a regular space-time the invariants are non-vanishing at the origin $r = 0$. For the invariant r_2 we find

$$r_2 = \left(8\pi\rho + 8\pi p - \frac{4m}{r^3}\right)^2 + 2\left(8\pi p - 8\pi\Lambda + \frac{2m}{r^3}\right)^2 + 2\left(8\pi\rho + 8\pi\Lambda - \frac{2m}{r^3}\right)^2 + 4\left(\frac{2m}{r^3}\right)^2. \quad (11)$$

For a monotonically decreasing and regular pressure and density functions, the function r_2 is also regular and monotonically decreasing throughout the star. Therefore it satisfies the condition $r_2(R) < r_2(0)$. By assuming that the surface density is vanishing, $\rho(R) = \rho_{surface} = 0$, we obtain the following general constraint upon the mean density of the star:

$$\left(\frac{2M}{R^3}\right)^2 + 2\left(\frac{M}{R^3} - 4\pi\Lambda\right)^2 < 8\pi^2 \left[\left(p_c + \frac{\rho_c}{3}\right)^2 + 2\left(p_c + \frac{\rho_c}{3} - \Lambda\right)^2 + 2\left(\frac{2\rho_c}{3} + \Lambda\right)^2 + \frac{4}{9}\rho_c^2 \right]. \quad (12)$$

III. DISCUSSIONS AND FINAL REMARKS

The existence of a limiting value of the mass-radius ratio leads to upper bounds for other physical quantities of observational interest. One of these quantities is the surface red shift z , defined in the Schwarzschild-de Sitter geometry according to $z = \left(1 - \frac{2M}{R} - \frac{8\pi}{3}\Lambda R^2\right)^{-\frac{1}{2}} - 1$.

In the absence of a cosmological constant Eq.(9) leads to the well-known constraint $z \leq 2$. For $\Lambda \neq 0$ the surface red shift must obey the general restriction

$$z \leq \frac{3}{1 - \frac{2\Lambda}{\bar{\rho}}} - 1. \quad (13)$$

As another application of the obtained upper mass -radius ratios we shall derive an explicit limit for the total energy of the compact general relativistic star. The total energy (including the gravitational field contribution) inside an equipotential surface S can be defined to be [14]

$$E = E_M + E_F = \frac{1}{8\pi}\xi_s \int_S [K] dS, \quad (14)$$

where ξ^i is a Killing field of time translation, ξ_s its value at S and $[K]$ is the jump across the shell of the trace of the extrinsic curvature of S , considered as embedded in the 2-space $t = constant$. $E_M = \int_S T_i^k \xi^i \sqrt{-g} dS_k$ and E_F are the energy of the matter and of the gravitational field, respectively. This definition is manifestly coordinate invariant. In the case of a static spherically symmetric matter distribution from Eq. (14) we obtain the

following exact expression: $E = -re^{\frac{\nu-\lambda}{2}}$ [14]. Hence the total energy of a compact general relativistic object in a Schwarzschild-de Sitter space-time is $E = -R \left(1 - \frac{2M}{R} - \frac{8\pi}{3}\Lambda R^2\right)$.

For $\Lambda = 0$ we find the following upper limit for the total energy of the star: $E \leq -\frac{R}{9}$. In the presence of a cosmological constant we have

$$E \leq -\frac{R}{9 \left(1 - \frac{2\Lambda}{\rho}\right)^2} \quad (15)$$

In the present Letter we have considered the mass-radius ratio bound for compact general relativistic objects in the more general Schwarzschild- de Sitter geometry. Also in this case it is possible to obtain explicit inequalities involving $\frac{2M}{R}$ as an explicit function of the cosmological constant Λ . The surface red shift and the total energy (including the gravitational one) are modified due to the presence of the extra term in the gravitational field equations.

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